Generalized $\Gamma$-Cancellativity of $\Gamma$-AG-Groupoids

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Abstract-In this paper we study some properties of a $\Gamma$-cancellativity on a $\Gamma$-AG-groupoid. Finally we study quasi- $\Gamma$-cancellativity which is a generalization of $\Gamma$-cancellativity.

Keywords $\Gamma$-AG-groupoid, $\Gamma$-cancellativity, quasi-$\Gamma$-cancellativity

1. Introduction

Definition 1.1 [1. P.41]. A groupoid $(S, \cdot)$ is called an AG-groupoid, if it satisfies left invertive law

$(ab)c = (cb)a$ for all $a, b, c \in S$.

Lemma 1.2 [1. P.41]. An AG-groupoid $S$, is called a medial law if it satisfies

$(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$.

Definition 1.3 [8. P.110]. An AG-groupoid $S$, is called a paramedical if it satisfies

$(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in S$.

Proposition 1.4 [2. P.110]. If $S$ is an AG-groupoid with left identity, then

$a(bc) = b(ac)$ for all $a, b, c, d \in S$.

Definition 1.5. [8, p.268] Let $S$ and $\Gamma$ be any non-empty sets. We call $S$ to be $\Gamma$-AG-groupoid if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written $(a, \alpha, b)$ by $a\alpha b$, such that $S$ satisfies the identity $(a\alpha b)\beta c = (a\alpha b)\beta a$ for all $a, b, c, \alpha, \beta \in \Gamma$.

Definition 1.6. [4, p.2]. Let $S$ and $\Gamma$ be any non-empty sets. If there exists a mapping $S \times \Gamma \times S \rightarrow S$, written $(a, \alpha, b)$ by $a\alpha b$, $S$ is called a $\Gamma$-medial if it satisfies $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$ and called a $\Gamma$-paramedial if it satisfies $(a\alpha b)\beta(c\gamma d) = (d\alpha b)\beta(c\gamma d)$ for all $a, b, c, d \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

Shan M. was introduced the concepts of cancellativity and quasi-cancellativity of an AG-groupoids as follows.

Definition 1.7. [7. P2188]. An element $a$ of a AG-groupoid $S$ is called left cancellative if $ax = ay$ implies that $x = y$ for all $x, y \in S$. Similarly an element $a$ of a AG-groupoid $S$ is called right cancellative if $xa = ya$ implies that $x = y$ for all $x, y \in S$. An element $a$ of an AG-groupoid $S$ is called cancellative if it is both left and right cancellative.

Definition 1.8 [6. P2066]. An AG-groupoid $S$ is a quasi-cancellative if for any $x, y \in S$,

1. $x = xy$ and $y^2 = yx$ implies that $x = y$,
2. $x = yx$ and $y^2 = xy$ implies that $x = y$. 


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2. \( \Gamma \)-Cancellativity of \( \Gamma \)-AG-groupoids

In this paper, we introduce the concept of a \( \Gamma \)-cancellativity of \( \Gamma \)-AG-groupoids which is defined analogous to [6] and investigate its properties.

**Definition 2.1.** An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called left \( \Gamma \)-cancellative if \( aax = aay \) implies that \( x = y \) for all \( x, y \in S \) and \( a \in \Gamma \). Similarly an element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called right \( \Gamma \)-cancellative if \( xaa = yaa \) implies that \( x = y \) for all \( x, y \in S \) and \( a \in \Gamma \). An element \( a \) of a \( \Gamma \)-AG-groupoid \( S \) is called \( \Gamma \)-cancellative if it is both left and right \( \Gamma \)-cancellative.

**Theorem 2.2.** The following statements are equivalent for a \( \Gamma \)-AG-groupoid \( S \):
1. \( S \) is left \( \Gamma \)-cancellative;
2. \( S \) is right \( \Gamma \)-cancellative;
3. \( S \) is \( \Gamma \)-cancellative.

**Proof.** (1) \( \Rightarrow \) (2) Let \( S \) be left \( \Gamma \)-cancellative. Let \( a \) be an arbitrary element of \( S \) and let \( xaa = yaa \). Let \( k \in S \) and \( \beta \in \Gamma \) for all \( x, y \in S \) and \( a \in \Gamma \). Then
\[
(kaa)b\beta x = (xa\alpha)\beta y \\
= (\alpha\alpha)b\beta y
\]
by Definition 1.1.

By left \( \Gamma \)-cancellativity, \( x = y \). Thus \( S \) is right \( \Gamma \)-cancellative.

(2) \( \Rightarrow \) (3) Let \( S \) be right \( \Gamma \)-cancellative. Let \( a \) be an arbitrary element of \( S \) and let \( aaax = aay \).

Let \( k \in S \) and \( \beta \in \Gamma \) for all \( x, y \in S \) and \( a \in \Gamma \). Then
\[
[(xbk)\alpha\alpha]y\gamma a = (a\alpha\alpha)y(xbk) \\
= (a\alpha\alpha)y(a\beta k)
\]
by Definition 1.1.

By right \( \Gamma \)-cancellativity, \( x = y \). Thus \( S \) is left \( \Gamma \)-cancellative. Hence \( S \) is \( \Gamma \)-cancellative.

(3) \( \Rightarrow \) (1) This is clear.

**Theorem 2.3.** Every right \( \Gamma \)-cancellative element of a \( \Gamma \)-AG-groupoid \( S \) is a left \( \Gamma \)-cancellative.

**Proof.** Let \( S \) be a \( \Gamma \)-AG-groupoid and let \( a \) be an arbitrary right \( \Gamma \)-cancellative element of \( S \). Suppose that \( a\alpha x = a\alpha y \) for all \( a, x, y \in S \) and \( a \in \Gamma \). For \( \beta, \gamma \in \Gamma \), we have
\[
[(xb\alpha)\alpha\alpha]y\gamma a = (a\alpha\alpha)y(xb\alpha) \\
= (a\alpha\alpha)y(a\beta\alpha)
\]
by Definition 1.1.

Thus the right \( \Gamma \)-cancellativity implies that \( x = y \). Hence \( a \) is left \( \Gamma \)-cancellative. Therefore every right \( \Gamma \)-cancellative element of \( S \) is left \( \Gamma \)-cancellative.

**Definition 2.4.** [8, p269] An element \( e \in S \) is called a left identity of a \( \Gamma \)-AG-groupoid if \( eya = a \) for all \( a \in S \) and \( \gamma \in \Gamma \).
The following two theorems are analogously to the in [7, p.2190].

**Theorem 2.5.** Let \( S \) be a \( \Gamma \)-AG-groupoid with a left identity \( e \) which is right \( \Gamma \)-cancellative. If \( aab = cad \), then \( b\gamma a = d\gamma c \) for all \( a, b, c, d \in S \) and \( \alpha, \gamma \in \Gamma \).

**Proof.** Let \( a, b, c, d \in S \) and \( \alpha, \gamma \in \Gamma \). Then by Definitions 1.1 and 2.4, we have the following implication

\[
aab = cad \Rightarrow (e\gamma a)ab = (e\gamma c)ad \Rightarrow (b\gamma a)\alpha e = (d\gamma c)\alpha e.
\]

Since \( S \) is right \( \Gamma \)-cancellative, thus \( b\gamma a = d\gamma c \). We complete the proof. □

**Theorem 2.6.** Let \( S \) be a \( \Gamma \)-AG-groupoid with a left identity \( e \). Then every left \( \Gamma \)-cancellative element is also right \( \Gamma \)-cancellative.

**Proof.** Let \( a \) be an arbitrary left \( \Gamma \)-cancellative element of \( S \) and suppose that \( xaa = yaa \) for all \( x, y \in S \) and \( \alpha \in \Gamma \). Then by Theorem 2.5, we have \( aax = aay \). Since \( S \) is left \( \Gamma \)-cancellative, \( x = y \). Thus \( a \) is right \( \Gamma \)-cancellative. Hence every left \( \Gamma \)-cancellative element of \( S \) is right \( \Gamma \)-cancellative. □

A left invertible property of a \( \Gamma \)-AG-groupoid is defined analogously to AG-groupoid as in [5,p387].

**Definition 2.7.** Let \( S \) be a \( \Gamma \)-AG-groupoid with a left identity \( e \). An element \( a \) of \( S \) is said to be left invertible if there exists an element \( a^{-1} \) of \( S \) such that \( a^{-1}aa = e \) for all \( \alpha \in \Gamma \). In this case \( a^{-1} \) is called a left inverse of \( a \). Dually, an element \( a \) of \( S \) is said to be right invertible if there exists an element \( a^{-1} \) of \( S \) such that \( aa^{-1}a = e \) for all \( \alpha \in \Gamma \), \( a^{-1} \) is called a right inverse of \( a \). If an element \( a \) of \( S \) is both left and right invertible, then \( a \) is called invertible.

Next, we prove that cancellativity and invertibility are coincident in a finite \( \Gamma \)-AG-groupoid \( S \) with a left identity \( e \).

**Theorem 2.8.** Let \( S \) be a finite \( \Gamma \)-AG-groupoid \( S \) with a left identity \( e \), then for all \( a \in S \), \( a \) is invertible if and only if \( a \) is \( \Gamma \)-cancellative.

**Proof.** \((\Rightarrow)\) Assume that \( a \) is invertible. Then there exists \( a^{-1} \in S \) such that \( a^{-1}aa = e = a\alpha a^{-1} \). Suppose that \( xaa = yaa \) for all \( x, y \in S \) and \( \gamma \in \Gamma \). Then

\[
x = e\gamma x = (a^{-1}aa)yx = (xaa)\gamma a^{-1} = (yaa)\gamma a^{-1} = (a^{-1}aa)\gamma y = e\gamma y = y
\]

Thus \( a \) is right \( \Gamma \)-cancellative. By Theorem 2.2, \( a \) is \( \Gamma \)-cancellative.

\((\Leftarrow)\) Assume that \( a \) is \( \Gamma \)-cancellative and let \( S = \{s_1, s_2, \ldots, s_n\} \). Then for all \( \alpha \in \Gamma \),

\[
aas_1, aas_2, \ldots, aas_n \]

are all distinct. Since \( S \) is finite, there must exists a positive integer \( i \in \{1, 2, \ldots, n\} \) such that \( aas_i = e \) but then \( s_i\alpha a = e \). By Theorem 2.5, we have \( aas_i = s_i\alpha a = e \) for all \( \alpha \in \Gamma \). Hence \( a \) is invertible. □

**3. The Quasi-\( \Gamma \)-Cancellativity of a \( \Gamma \)-AG-groupoid**

In section, we study definition of a quasi-\( \Gamma \)-cancellativity which is defined analogously as in [6, P2066] and also investigate its properties.

**Definition 3.1.** A \( \Gamma \)-AG-groupoid \( S \) is a quasi-\( \Gamma \)-cancellative if for any \( x, y \in S \) and \( \gamma \in \Gamma \),
(1) \( x = x y y \) and \( y = y y x \) implies that \( x = y \),
(2) \( x = y y x \) and \( y = x y y \) implies that \( x = y \).

**Definition 3.2.** A \( \Gamma \)-AG-groupoid \( S \) is said to be a \( \Gamma \)-idempotent. If \( x y x = x \) for all \( x \in S \) and \( \gamma \in \Gamma \).

**Definition 3.3.** A \( \Gamma \)-AG-groupoid \( S \) is said to be a \( \Gamma \)-AG-band if every element of \( S \) is a \( \Gamma \)-idempotent.

The following two theorems are analogously as in [6, p.2067-2068].

**Theorem 3.4.** Every \( \Gamma \)-AG-band is a quasi-\( \Gamma \)-cancellative.

**Proof.** Let \( S \) be a \( \Gamma \)-AG-band and let \( x, y \in S \) and \( \gamma, \beta \in \Gamma \). We shall show that \( S \) is a quasi-\( \Gamma \)-cancellative consider the following:

1. Assume that \( x = x y y \) and \( y = y y x \). Then \( x = x y y \) and \( y = y y x \). Now
   \[
   x = x y y \\
   = (x y x) \beta y \\
   = (y y x) \beta x \\
   = y \beta x \\
   = x y y \\
   = \gamma y x = x
   \]
   by Definition 3.2
   by Definition 1.1
   by \( \Gamma \)-paramedical
   by \( \gamma \in \Gamma \).

2. Assume that \( x = y y x \) and \( y = x y y \). Then \( x = y y x \) and \( y = x y y \). Now
   \[
   x = y y x \\
   = (y y x) \beta (x y x) \\
   = (x y x) \beta (y y y) \\
   = x y \beta y \\
   = x y y \\
   = \gamma y y = x
   \]
   by Definition 3.2
   by \( \gamma \in \Gamma \).

**Theorem 3.5.** Let \( S \) be a \( \Gamma \)-AG-groupoid such that \( S \) is quasi-\( \Gamma \)-cancellative and \( x y x = x \) for all \( x \in S \) and \( \gamma \in \Gamma \). If \( S \) is \( \Gamma \)-medial, then the following statements hold:

1. \( x y a = x \gamma b \) if and only if \( a y x = b y x \),
2. \((x y y) \beta a = (x y y) \beta b \) implies that \( a \beta (y y x) = b \beta (y y y) \), for all \( x, y, a, b \in S \) and \( \gamma, \beta \in \Gamma \).

**Proof.** (1) \(( \Rightarrow \) Let \( x y a = x \gamma b \). Then \( (x y a) \beta (x y a) = (x y b) \beta (x y a) \) and \( (x y a) \beta (x y a) = (x y b) \beta (x y b) \). So
   \[
   a \beta x = (a y a) \beta x \\
   = (x y a) \beta a \\
   = (x y b) \beta a \\
   = (a \gamma b) \beta x \\
   = (a \gamma b) \beta (x y x) \\
   = (a \gamma x) \beta (b y x) \\
   = \gamma (x y y) = x
   \]
   by Definition 3.2
   by Definition 1.1
   by \( \gamma \in \Gamma \).

And
   \[
   b \beta x = (b \gamma b) \beta x \\
   = (x y b) \beta b \\
   = \gamma y y = x
   \]
   by Definition 3.2
   by Definition 1.1
Then \( a\bar{b}x = (a\gamma x)\beta(b\gamma x) \) and \( b\beta x = (b\gamma x)\beta(a\gamma x) \). Thus \( a\gamma x = b\gamma x \).

(\(\Leftarrow\)) This can be proved similarly.

\[ \begin{aligned}
(2.) & \quad \text{Let } x, y, a, b \in S \text{ and } \gamma, \beta \in \Gamma \text{ such that } (x\gamma y)\beta a = (x\gamma y)\beta b. \text{ Then } a = b. \text{ So we have} \\
& \quad (x\gamma y)\beta a = (x\gamma y)\beta (a\gamma a) = (x\gamma y)\beta (a\gamma b) \\
& \quad (y\gamma x)\beta a = (y\gamma x)\beta (a\gamma a) = (y\gamma x)\beta (a\gamma b) \\
& \quad \Rightarrow a\beta (y\gamma x) = [a\beta (y\gamma x)]a[a\beta (y\gamma x)] = [b\beta (y\gamma x)]a[b\beta (y\gamma x)] \tag{3.1}
\end{aligned} \]

Similarly, if \((x\gamma y)\beta a = (x\gamma y)\beta b\), then \( a = b \), and so

\[ \begin{aligned}
& \quad b\beta (x\gamma y) = (byb)\beta (x\gamma y) = (bya)\beta (x\gamma y) \\
& \quad (x\gamma y)\beta b = (x\gamma y)\beta (byb) = (x\gamma y)\beta (bya) \\
& \quad (y\gamma x)\beta b = (y\gamma x)\beta (byb) = (y\gamma x)\beta (bya) \\
& \quad (a\beta \gamma (y\gamma x) = (byb)\beta (y\gamma x) = b\beta (y\gamma x) \\
& \quad \Rightarrow b\beta (y\gamma x) = [b\beta (y\gamma x)]a[b\beta (y\gamma x)] = [a\beta (y\gamma x)]a[b\beta (y\gamma x)] \tag{3.2}
\end{aligned} \]

From (3.1), and (3.2) we have \( a\beta (y\gamma x) = b\beta (y\gamma x) \).

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**References**


